Venue ____________________

Student Number |__|__|__|__|__|__|__|__|

Research School of Economics

EXAMINATION

Semester 1 - End of Semester Deferred/Supplementary, 2024

ECON2125/ECON6012_Semester 1 Optimisation for Economics and Financial Economics

This paper is for ANU students.

Examination Duration: 180 minutes
Reading Time: 15 minutes
Exam Conditions:
N/A
Materials Permitted In The Exam Venue: no materials permitted
(No electronic aids are permitted e.g. laptops, phones)
Materials To Be Supplied To Students: 1 x 20 page
Instructions To Students: See next page

INSTRUCTIONS TO STUDENTS

- Read the questions carefully.
- Questions are worth different amount of marks given in parenthesis. Sub-questions in each questions are of equal value unless marked otherwise.
- To maximize your marks, explain all the steps in your arguments.
- If any part of the question seems missing or ambiguous, state clearly the way you interpret it, and carry on with your answer.
- In solving the questions, you can use any fact from the lecture materials without proof, unless specifically asked to give details. In either case, you should clearly state the relevant fact.
- You do not need to do the questions in order, as long as you clearly mark in your answer sheets which question you are addressing.
- Do not forget to write your ANU student number on the title page.

QUESTIONS

Question 1. (*10 marks in total*)

Which values of λ make the following matrix singular (non-invertible)?

$$\begin{pmatrix} 1 & \lambda & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 \\ \frac{6}{3024} & \frac{5}{3024} & \frac{4}{3024} & 2 & 1 \\ \frac{3}{3024} & \frac{2}{3024} & \frac{1}{3024} & -\lambda & 1 \end{pmatrix}$$

Question 2. (*10 marks*)

A software engineer is assigned to refactor a piece of code composed of two parts. They have to allocate no more than the total time budget T for the work on the assignment. The time t_i, $i = 1, 2$, spent on each part of the code increases the probability for it to work correctly according to the formula $p_i = 1 - \alpha_i \exp(-t_i)$, where α_1 and α_2 are known constants. The engineer tries to maximize the log of the probability of successful refactoring given by $\log(p_1 p_2)$.

(a) Write down the corresponding optimization problem and name all of its essential parts.

(b) Find the (candidate) solution of the problem using first order conditions only.

Question 3. (*15 marks*)

Using the definition of convergence, show that if a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is continuous in a point $a \in \mathbb{R}^N$, then the function $g(x) = \exp\big(f(x)\big)$ is also continuous at this point.

Question 4. (*15 marks in total*)

For each of the following optimization problems determine whether it has a solution; if so determine if the solution is unique.

Hint: you don't have to solve any of the problems, though solving a problem is one way to determine the number of solutions

(a) $max_{x,y}(x^2 + y^2)$ s.t. $x^4 + y \leqslant 10, x - 2y \leqslant 0$

(b) $max_{x,y,z}xyz$ s.t. $x^4 + y \leqslant 10, x - 2y \leqslant 0$

(c) $max_{x,y,z}(x \ln x - x \ln y - z^{-2})$ s.t. $3x + y + 2z \leqslant 107, x > 0, y > 0, z > 0$

Question 5. (*15 marks*)

Consider a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by the formula $f(x_1, x_2, x_3) = \exp(x_1)x_2 \ln(|x_3|)$.

Derive the approximation of this function around the point $(1, 1, 1)$ using the gradient and the Hessian of f.

Question 6. (*15 marks*)

Consider a recursive optimization problem in $t = 1, \dots, T$

$$V_t(M) = \begin{cases} \max_{x \in D} \left[u(x) + V_{t+1}(M - x)\right] & t < T \\ \max_{x \in D} u(x) & t = T \end{cases}$$

where $D \subset \mathbb{R}$ is closed and bounded, and $u\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Prove by induction that the maximizer exists for each $t \in \{1, \dots, T\}$, taking as given the following fact.

Maximum theorem (special case thereof): In any optimization problem $V(\theta) = \max_{x \in D} f(x, \theta)$ where $f(x, \theta)\colon \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ is a continuous function and D is a compact set, $V(\theta)$ is itself a continuous function of θ.

Question 7. (*30 marks*)

Consider the following constrained optimization problem which represents a case of utility maximization under non-linear budget constraint

$$\begin{aligned} &\max_{x_1, x_2} \{\alpha \log(x_1) + (1 - \alpha) \log(x_2)\} \\ &\quad \text{s.t.} \\ &\min(2x_1 + x_2;\ x_1 + 3x_2) \leqslant 10 \\ &x_1 \geqslant 0,\ x_2 \geqslant 0 \end{aligned}$$

where $\alpha \in (0, 1)$

(a) (*5 marks*) Find the stationary point for $\alpha = 1/2$

(b) (*5 marks*) Confirm the found point is indeed a maximizer by verifying the second order condition

(c) (*20 marks*) Find the values of α for which the solution of the problem lies on the first of second segment of the budget line (the first or the second inequality constraint). What is the solution of the problem (if it exists) for all other values of α?

Question 8. (*30 marks*)

Find local minimizers and maximizers of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = y^2 + 2x^2$ on the set

$$D = \{(x,y)\colon \frac{x^2}{9} + y^2 \leqslant 5 \text{ and } x^2 - y^2 \geqslant 5\}.$$

Use first and second order conditions to support your answer.

END OF EXAMINATION